A Dynamic Programming Approach for the Fuzzy Minimum Cost Maximum Flow Problem in the Multi-Stage Networks

Nermin Kartli
Ankara University – Department of Computer Science
E-posta: nkartli@ankara.edu.tr

Abstract

In this study, we consider the minimum-cost maximum-flow problem in a fuzzy environment for the multi-stage networks. In the problem under investigation, we model edge costs on the network with parametric fuzzy numbers. We transform uncertainty into deterministic subproblems using the α-cut approach and propose a novel dynamic-programming-based heuristic algorithm.

In the proposed method, a dynamic-programming-based shortest-path algorithm, developed to accommodate the multi-stage network structure, is applied to the deterministic problem obtained for each α level. In this process, performed on the residual network, the capacity dominance criterion is considered alongside cost minimization, and paths with higher flow capacity are preferred among alternative paths with the same cost. This approach ensures a more balanced and stable solution process.

The algorithm updates the residual network at each increment step and defines reverse edges with negative costs. This results in an iterative improvement process similar to classical minimum cost flow methods. Left and right end solutions are calculated separately, the results are combined, and defuzzification is applied in accordance with fuzzy decision theory.

The effectiveness of the proposed algorithm was evaluated through numerical experiments performed on randomly generated multi-stage test networks. The results show that the developed method can produce consistent, stable, and computationally efficient solutions under different uncertainty levels. In this respect, the study offers a practical and applicable approach to solving fuzzy network flow problems.

Keywords: Fuzzy optimization, maximum flow at minimum cost, multi-stage networks, dynamic programming, alpha-cuts method, parametric fuzzy numbers, supply chain networks, heuristic algorithms.

1 Introduction

Network-based optimization problems are widely used in many fields such as logistics, transportation, production planning, supply chain management, telecommunications, and energy systems. Among these problems, the minimum cost maximum flow problem stands out as a fundamental model aiming to send the highest possible flow from a given source to a given destination at the lowest cost. In the classical case, this problem can be efficiently solved using linear programming techniques and specialized network algorithms.

However, in real-life applications, network parameters are often not precisely known and contain various uncertainties. Many external factors, such as transportation costs, fuel prices, exchange rates, demand fluctuations, weather conditions, and operational risks, cause system parameters to change over time. Classical models that do not consider such uncertainties may not adequately reflect the actual system behavior in practical applications and can produce misleading results for decision-makers.

Fuzzy set theory is widely used in modeling optimization problems defined under uncertainty. Fuzzy numbers allow for the representation of uncertain parameters without the need for probability distributions. Therefore, numerous studies have been conducted on network flow problems defined in a fuzzy environment in recent years. However, a significant portion of existing studies offer solutions based on classical shortest path algorithms on general network structures and do not adequately utilize the structural features of the problem.

Especially in layered structures such as multi-stage supply chain networks, classical shortest path algorithms lead to unnecessary computations and significantly increase the solution time. In such networks, nodes are divided into specific stages, and flow occurs only between successive stages. The effective use of this structure in the solution process is of great importance in terms of computational efficiency.

Although various alpha-cut approaches have been developed in the literature for fuzzy minimum cost maximum flow problems, most of these methods are not specifically designed for multi-stage network structures. Furthermore, capacity differences between alternative paths with the same cost are generally ignored in existing methods, which can negatively affect the total flow volume.

In this study, a new dynamic programming-based heuristic algorithm specific to multi-stage networks is proposed for solving the fuzzy minimum cost maximum flow problem. In the proposed method, fuzzy edge costs are modeled using a parametric representation and transformed into deterministic subproblems using the α-cut approach. For each α level, a special shortest path algorithm based on dynamic programming, utilizing the hierarchical structure of the network, has been developed.

Furthermore, by applying a capacity dominance criterion among alternative paths with the same cost, paths with higher flow potential are preferred. This aims to minimize the total cost and increase the amount of flow sent. The solution process is systematically improved by iteratively updating the network structure.

The main contributions of this study can be summarized as follows:

- A new dynamic programming-based solution method is proposed for the fuzzy minimum cost maximum flow problem defined in multi-stage networks.
- Parametric fuzzy costs are efficiently solved by transforming them into deterministic problems using the α-cut approach.
- More balanced and stable augmenting paths are obtained by using the capacity dominance criterion. The effectiveness of the proposed algorithm has been demonstrated through numerical experiments performed on randomly generated test networks.

The rest of the article is structured as follows: In the second section, we provide a summary of previous studies related to this study. The third section describes the problem in detail. The fourth section presents the proposed algorithm. The fifth section discusses the computational experiments and the results obtained. The final section provides general evaluations and suggestions for future research.

2 Related Work

Minimum cost flow problems are one of the fundamental problems that have long been studied in the field of classical network optimization. However, modeling cost and capacity parameters in networks with uncertainty makes the direct application of classical methods difficult. Studies on fuzzy minimum cost flow problems in the literature have addressed uncertainty models from different perspectives.

From a fundamental perspective, Mares and Horak (1983) [13] have defined fuzzy quantities in networks, and then Keyi and Wang [12] have considered the fuzzy minimum-cost flow problem in a network. These studies are the first to address the effect of uncertain costs on network flow problems.

Ghatee and Hashemi have conducted research within the context of extended fuzzy flow models to address the fuzzy supply-demand balance and propose different solution methods. They have proposed the method within the scope of fuzzy multi-commodity flow, aiming to optimize the transportation of multiple products simultaneously under flow and cost uncertainty [10].

The study presented by Bozhenyuk and Gerasimenko (2013) [2] addresses the minimum cost flow problem in fuzzy dynamic networks and proposes a solution method based on the fuzzification of parameters such as sub-stream bounds, transit cost, and time. This study focuses on integrating fuzzy costs and transit times into the flow process.

Similarly, Bozhenyuk et al. [3] presented a method for determining minimum cost flow in fuzzy dynamic networks using substream bounds and costs. This study has developed approaches to flow problems in uncertain environments where dynamic parameters can change over time.

In the study by El-Sherbeny (2019) [8], a model that fuzzifies time windows as a valid extension of the classification problem is proposed, and an algorithm suitable for this model is developed. This study presents a path-based solution using array-by-array augmenting by combining fuzzy costs and time windows.

In the study [6] by Chung and Duan, multi-stage fuzzy neural network (MSFNN) models were developed to overcome the "curse of dimensionality" problem encountered by single-stage fuzzy neural networks in high-dimensional problems. The authors divided the fuzzy inference process into multiple stages by defining two hierarchical network structures: incremental and aggregated. This structure prevents the exponential increase of the number of fuzzy rules and provides a more efficient modeling approach. In addition, structural and parameter learning algorithms of the proposed models were developed, and their effectiveness was demonstrated through various numerical experiments. This work focuses on prediction and modeling problems by addressing multi-stage fuzzy inference within the framework of neural networks; it is not directly related to network flows or optimization-based decision problems. In contrast, our work develops a dynamic programming-based optimization algorithm for solving the minimum cost maximum flow problem in a fuzzy environment and focuses on the decision-making process on network structures.

Ding et al. [7] transform the target controllability problem with maximum covered nodes set in multiplex networks into a minimum-cost maximum-flow problem based on graph theory. They have proposed an algorithm called target minimum-cost maximum-flow (TMM).

In his study, Baidya [1] modeled the uncertainty in multi-stage supply chain networks using grey numbers and solved the problem by transforming it into deterministic sub-models. In the study, the objective function in interval form was made single-objective using the mean-width method, and the models were solved with the help of LINGO.

The studies [4, 5] propose a novel algorithmic framework capable of solving minimum-cost and maximum-cost flow problems in near linear time. The method is based on a combination of interior point methods and dynamic graph data structures, updating the flow incrementally over approximate minimum-rate cycles. This yields one of the fastest known theoretical solutions for very large-scale and deterministic networks. However, in this study, edge costs and capacities are treated as exact (deterministic), and uncertainty or fuzzy structures are not considered. Furthermore, the algorithm is designed for generalized directed networks and does not include multi-stage specialized network structures or α-cut-based fuzzy solution approaches.

In the study [14], decision problems on multi-stage and uncertain network structures are addressed within the framework of fuzzy set theory. Edge costs and system parameters are modeled with fuzzy numbers, and uncertainties are transformed into deterministic subproblems using the α-cut approach. The resulting subproblems are solved using classical network optimization methods, and the results are reduced to a single decision value using defuzzification techniques.

This study has conceptual similarities with the proposed method in that it offers a systematic framework for modeling and solving multi-stage networks in a fuzzy environment. While the aforementioned study focuses on modeling fuzzy parameters in multi-stage networks and creating a general solution framework, in this study, we propose a dynamic programming-based shortest path algorithm specifically developed for multi-stage network structures. Furthermore, while the current study uses general-purpose classical network solution methods, the proposed approach considers the network structure and capacity dominance criteria together, offering a more efficient solution mechanism specific to the fuzzy minimum cost maximum flow problem.

3 Problem Statement

In this section, we define the fuzzy minimum cost maximum flow problem on a multi-stage directed network.

3.1 Network Structure

We assume that a directed network $G = (V, E)$ is given, where V represents the set of nodes and E represents the directed edge set. We also assume that the considered network can be divided into $m + 1$ stages as follows:

$$V = V_0 \cup V_1 \cup \cdots \cup V_m \tag{1}$$

Here,

- $V_0 = \{S\}$ represents the source node,
- $V_m = \{T\}$ represents the destination node,
- $V_1, V_2, \ldots, V_{m-1}$ represent the intermediate stages

The network structure is defined such that there is only a connection between successive stages. Accordingly, the following condition is satisfied:

$$(i, j) \in E \Rightarrow i \in V_k, j \in V_{k+1}, \quad k = 0, 1, \ldots, m-1 \tag{2}$$

3.2 Capacities and Flow Variables

A positive capacity value is defined for each $(i, j) \in E$ edge:

$$c_{ij} > 0 \tag{3}$$

The flow variable x_{ij} represents the amount of flow sent through the (i, j) edge and satisfies the following condition:

$$0 \leq x_{ij} \leq c_{ij}, \quad \forall (i, j) \in E \tag{4}$$

The flow conservation condition for the nodes is defined as follows:

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} F, & i = S \\ -F, & i = T \\ 0, & \text{other cases} \end{cases} \tag{5}$$

Here, F represents the total flow amount sent from node S to node T.

3.3 Fuzzy Edge Costs

For each edge $(i, j) \in E$, the unit cost is defined as a parametric fuzzy number:

$$\tilde{w}_{ij} = (u_{ij}(r), v_{ij}(r)), \quad r \in [0, 1] \tag{6}$$

Here, the functions $u_{ij}(r)$ and $v_{ij}(r)$ satisfy the following properties:

- $u_{ij}(r)$ is a bounded, non-decreasing and left-continuous function,
- $v_{ij}(r)$ is a bounded, non-increasing and left-continuous function,
- $u_{ij}(r) \leq v_{ij}(r)$, $\forall r \in [0, 1]$.

This definition allows for the parametric modeling of cost uncertainty.

3.4 α-cuts-Based Deterministic Model

Fuzzy costs are transformed into deterministic problems using the α-cuts approach. For each $\alpha \in [0, 1]$, the edge costs are obtained as follows:

$$w_{ij}^L(\alpha) = u_{ij}(\alpha) \tag{7}$$

$$w_{ij}^R(\alpha) = v_{ij}(\alpha) \tag{8}$$

Accordingly, two separate deterministic problems are defined for each α level.

3.5 Objective Function

For each α level, the objective is to minimize the total cost among the solutions that provide the maximum flow rate for the left end side:

$$\min \sum_{(i,j)\in E} w_{ij}^L(\alpha) x_{ij} \tag{9}$$

$$\max F \tag{10}$$

Similarly, for the right end side, we solve the following problem:

$$\min \sum_{(i,j)\in E} w_{ij}^R(\alpha) x_{ij} \tag{11}$$

$$\max F \tag{12}$$

This objective function is considered in lexicographic order, first aiming for the minimum cost, then the maximum flow rate.

3.6 Definition of Fuzzy Solution

By combining the left and right end solutions obtained for each α level, the fuzzy minimum cost function is defined as follows:

$$\tilde{M}(r) = (M_L(r), M_R(r)), \quad r \in [0, 1] \tag{13}$$

Here, $M_L(r)$ and $M_R(r)$ represent the left and right end cost functions, respectively.

3.7 Assumptions

The following assumptions are accepted in this study:

- The mesh structure is multi-stage and does not contain cycles.
- Edge capacities are exact and positive.
- Edge costs are modeled with parametric fuzzy numbers.
- The flow is divisible.
- Each α level is solved independently.

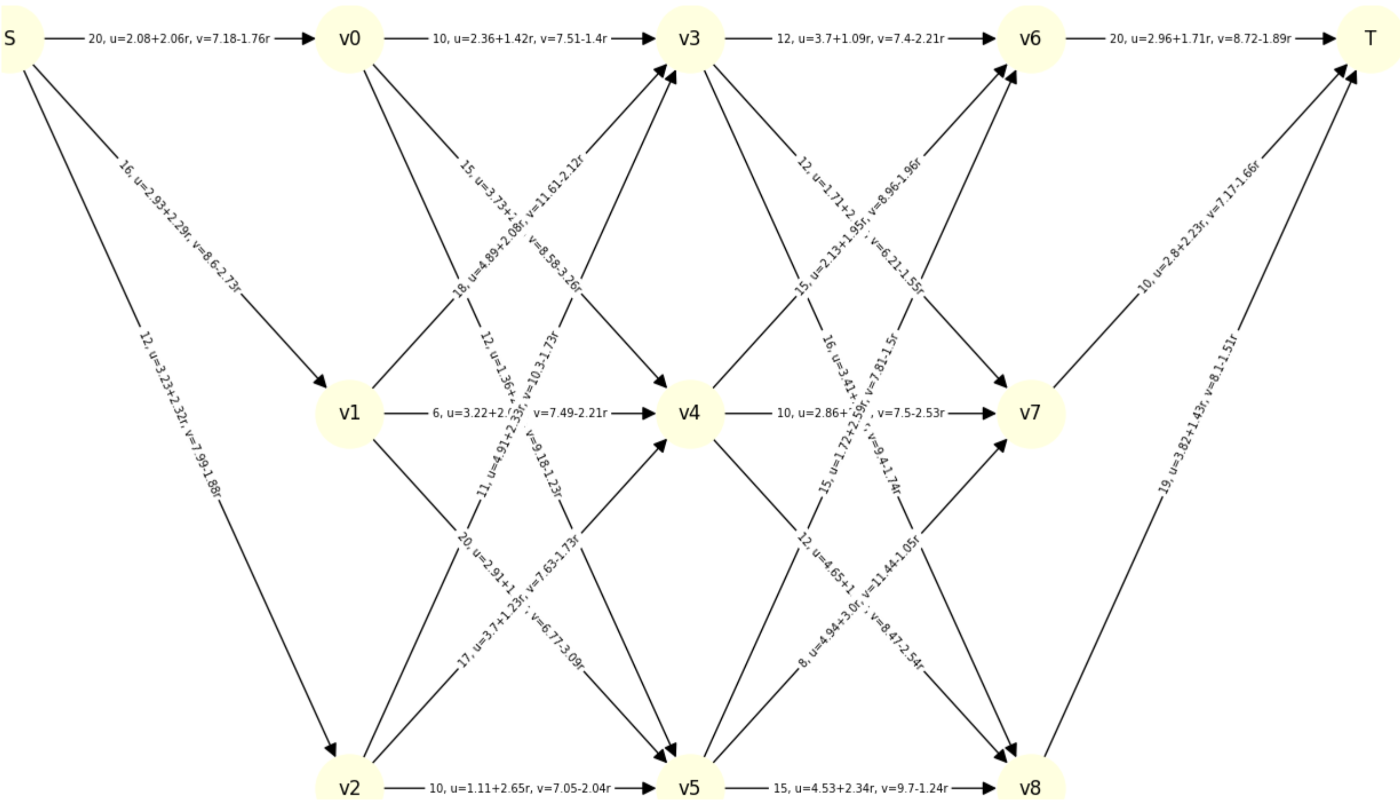

Figure 1: Fuzzy Minimum Cost Maximum Flow Problem:Example

4 Proposed Algorithm

We propose a new heuristic algorithm based on α-cuts and dynamic programming to solve the fuzzy minimum cost maximum flow problem defined on multi-stage supply chain networks.

The proposed method uses the α-cuts approach to reduce fuzzy costs to deterministic problems and solves a separate minimum cost maximum flow problem for each α level. The obtained solutions are then combined to form a fuzzy solution function. Thanks to this, fuzzy problems are transformed into numerous deterministic problems, as in the studies [9, 11].

The algorithm consists of two main components:

- α-cuts based fuzzy solution loop,
- Dynamic programming-based shortest path algorithm specific to the multi-stage network structure.

4.1 α-cuts Based Solution Approach

Fuzzy costs are defined in the form $(u_{ij}(r), v_{ij}(r))$ using a parametric representation. Here, $r \in [0,1]$, where each value of r represents a α-cut level.

In the algorithm, the interval $[0,1]$ is divided into N parts with equal steps, and the α levels are obtained as follows:

$$r_k = \alpha_k = \frac{k}{N}, \quad k = 0, 1, \dots, N \tag{14}$$

For each α value, the deterministic edge costs are calculated as follows:

- For the left end side, $w_{ij}^L(\alpha) = u_{ij}(\alpha)$,
- For the right end side, $w_{ij}^R(\alpha) = v_{ij}(\alpha)$.

In this way, a classic minimum cost maximum flow problem is obtained for each α level.

4.2 Residual Network and Flow Update Mechanism

For each α level, the residual network is initially defined as being the same as the original network.

The following steps are applied in each iteration:

1. The shortest path between S and T is calculated on the residual network.
2. The smallest capacity value on this path is determined as the amount of flow that can be increased.
3. The determined amount is sent along the path.
4. The capacities of the edges used are reduced.
5. The residual network is updated by adding reverse-directional edges.
6. The costs of reverse-directional edges are assigned as negative.

This process continues until there is no path left between S and T on the residual network.

4.3 Dynamic Programming Based Shortest Path Algorithm

The network is given in stages as follows:

$$V = V_0 \cup V_1 \cup \cdots \cup V_m \tag{15}$$

Here,

- $V_0 = \{S\}$,
- $V_m = \{T\}$,
- $V_1, \dots, V_{m-1}$ represent intermediate stages.

If there are multiple paths with equal cost, the capacity value is considered.

Let $cost[u, v]$ denote the unit cost of flow on edge (u, v), and let $c[u, v]$ denote the capacity of edge (u, v). For example, for the left end side and an α, $cost[u, v] = w_{ij}^L(\alpha) = u_{ij}(\alpha)$.

Let $M[i, v]$ represent the minimum unit cost of a path from the source node S to node v in stage i. Let $CA[i, v]$ denote the minimum capacity along the path corresponding to this minimum unit cost.

The initial conditions are defined as:

$$M[1, v] = cost[S, v], \quad CA[1, v] = c[S, v].$$

For each stage $i \geq 1$ and for each $v \in V_{i+1}$, the recursive relation is given by:

$$M[i+1, v] = \min_{u \in V_i} \{M[i, u] + cost[u, v]\}.$$

Let u^* be the node at which the minimum is attained. If there are multiple paths with equal cost, the capacity value is considered. In this case, we choose a node with maximum flow. Then, the corresponding capacity is updated as:

$$CA[i+1, v] = \min \{CA[i, u^*],\ c[u^*, v]\}.$$

4.4 Capacity Dominance Criterion

In the proposed method, when choosing between alternative paths with the same cost, not only the cost but also the minimum capacity value on the path is considered.

For two paths:

$$M_1 = M_2 \tag{16}$$

If this is the case, the following criterion is applied:

$$\max\{\min(CA_1, c_1), \min(CA_2, c_2)\} \tag{17}$$

In this way, the path with higher capacity is preferred.

This approach allows the algorithm to produce more balanced solutions.

4.5 Creating the Fuzzy Solution

After obtaining the left and right end solutions separately, the cost pairs are combined for each α level to create a fuzzy minimum cost function:

$$\tilde{M}(r) = (M_L(r), M_R(r)) \tag{18}$$

Here, $M_L(r)$ and $M_R(r)$ represent the left and right end cost functions, respectively.

If desired, the fuzzy solution can be reduced to a single deterministic value using defuzzification methods.

In this study, the weighted center method was used.

4.6 Advantages of the Algorithm

The main advantages of the proposed method are summarized below:

- It is specifically designed for multi-stage mesh structures.
- It has lower computational costs compared to the classic Bellman-Ford algorithm.
- It produces more stable solutions thanks to capacity dominance.
- It can directly take fuzzy costs into account.
- It has high applicability to large-scale problems.

Algorithm 1: Fuzzy Minimum-Cost Maximum-Flow Algorithm

Input : Directed multi-stage graph $G = (V, E)$,
Source node S, Sink node T,
Edge capacities c_{ij},
Fuzzy costs $\tilde{w}_{ij} = (u_{ij}(r), v_{ij}(r))$,
Number of α-levels N

Output: Fuzzy minimum cost $\tilde{M}(r)$,
Maximum flow F

Initialize $\alpha_k = k/N$, $k = 0, 1, \ldots, N$
Initialize result lists L, R
foreach *mode* $\in$ {*left*, *right*} **do**
    Set residual graph $R \leftarrow G$
    Set total flow $F \leftarrow 0$
    Set total cost $M \leftarrow 0$
    foreach $\alpha \in \alpha_k$ *(increasing for left, decreasing for right)* **do**
        Compute deterministic costs $w_{ij}(\alpha)$ from fuzzy costs
        Update edge weights in R using $w_{ij}(\alpha)$
        while *true* **do**
            $(P, f, m) \leftarrow$ DP_Shortest_Path(R, S, T)
            if $P = \emptyset$ **then**
                break
            Augment flow f on path P
            Update residual graph R
            Update $F \leftarrow F + f$
            Update $M \leftarrow M + f \cdot m$
        Store (α, F, M) in L or R
Construct fuzzy cost function $\tilde{M}(r)$ from L and R
return $\tilde{M}(r), F$

Algorithm 2: Dynamic Programming Shortest Path with Capacity Dominance

Input : Residual graph R,
Source S, Sink T,
Stage partition $\{V_0, V_1, \ldots, V_m\}$,
Edge weights w_{ij}, capacities c_{ij}

Output: Shortest path
Bottleneck capacity
Path cost

```
Initialize tables M[i][v] ← ∞, CA[i][v] ← 0, P[i][v] ← NULL
foreach v ∈ V_1 do
    if (S, v) ∈ E then
        M[1][v] ← w_Sv
        CA[1][v] ← c_Sv
        P[1][v] ← S
for i = 1 to m − 1 do
    foreach v ∈ V_{i+1} do
        best ← ∞
        bestCap ← 0
        foreach u ∈ V_i do
            if (u, v) ∈ E and M[i][u] < ∞ then
                val ← M[i][u] + w_uv
                cap ← min(CA[i][u], c_uv)
                if val < best then
                    best ← val
                    bestCap ← cap
                    P[i + 1][v] ← u
                else if val = best and cap > bestCap then
                    bestCap ← cap
                    P[i + 1][v] ← u
        M[i + 1][v] ← best
        CA[i + 1][v] ← bestCap
    if all M[i + 1][v] = ∞ then
        return ∅, 0, ∞
Reconstruct path p from table P
return p, CA[m][T], M[m][T]
```

4.7 Defuzzification Method

After obtaining the fuzzy minimum cost function $\tilde{M}(r) = (M_L(r), M_R(r))$ it needs to be converted into a single deterministic value that can be used in the decision-making process. For this purpose, the defuzzification process is applied.

In this study, the weighted centroid method, which is widely used in fuzzy decision theory, has been preferred. In this method, the fuzzy cost function is defuzzified as follows:

$$M^* = \frac{\int_0^1 \frac{M_L(r)+M_R(r)}{2} \cdot r\, dr}{\int_0^1 r\, dr} \tag{19}$$

If discrete α levels are used, this expression is approximately calculated in the following form:

$$C^* \approx \frac{\sum_{k=0}^{N} \frac{M_L(\alpha_k)+M_R(\alpha_k)}{2} \cdot \alpha_k}{\sum_{k=0}^{N} \alpha_k} \tag{20}$$

Here, N represents the number of levels α used.

This method produces a decision value that represents the uncertainty structure in a balanced way by considering both the left and right endpoints of the fuzzy cost function. It is also preferred because of its computationally simple nature and ease of implementation.

5 Computational Experiments and Results

In this section, the effectiveness and performance of the proposed algorithm are evaluated through numerical experiments conducted on randomly generated multi-stage networks. All experiments were performed in a simulation environment developed using the Python programming language.

5.1 Experimental Setup

Test problems were randomly generated considering different network sizes and number of stages. In each test network, nodes were assigned to specific stages, and edges were created such that connections only exist between consecutive stages.

The basic parameters used in the experiments are as follows:

- Number of stages: $m = 3, 4, 5$
- Number of nodes in each stage: Between 3 and 10
- Edge capacities: Random integers in the range $[5, 50]$
- Edge costs: Parametric fuzzy numbers
- Number of α levels: $N = 10$

Fuzzy cost functions were constructed using linear and triangular parametric structures. To ensure the reproducibility of the experiments, random number generation was performed using a fixed initial value (seed).

All experiments were conducted on a personal computer with an Intel i7 processor and 16 GB of RAM.

5.2 Evaluation Criteria

The performance of the proposed method was evaluated using the following criteria:

- Total maximum flow rate sent,
- Total fuzzy cost value,
- Defuzzified cost value,
- Solution time,
- Stability under different α levels.

In addition, the obtained results were compared with classical minimum cost maximum flow algorithms commonly used in the literature.

5.3 Experimental Results

For each test problem, left and right end solutions were calculated separately, and the resulting fuzzy cost functions were created. The results obtained for an example test problem are presented in Table 1.

Table 1: Experimental Results for a Sample Test Problem

α	Flow	Left End Cost	Right End Cost
0.0	48	525.83	1114.47
0.2	48	607.16	1201.67
0.4	48	687.43	1278.98
0.6	48	768.86	1356.14
0.8	48	843.71	1433.03
1.0	48	923.74	1510.85

When the results are examined, it is observed that the maximum flow rate remains constant across different α levels, while the total cost varies with the level of uncertainty. This shows that the proposed algorithm produces stable and consistent solutions.

Figure 2 shows the left and right end cost curves obtained for different α levels.

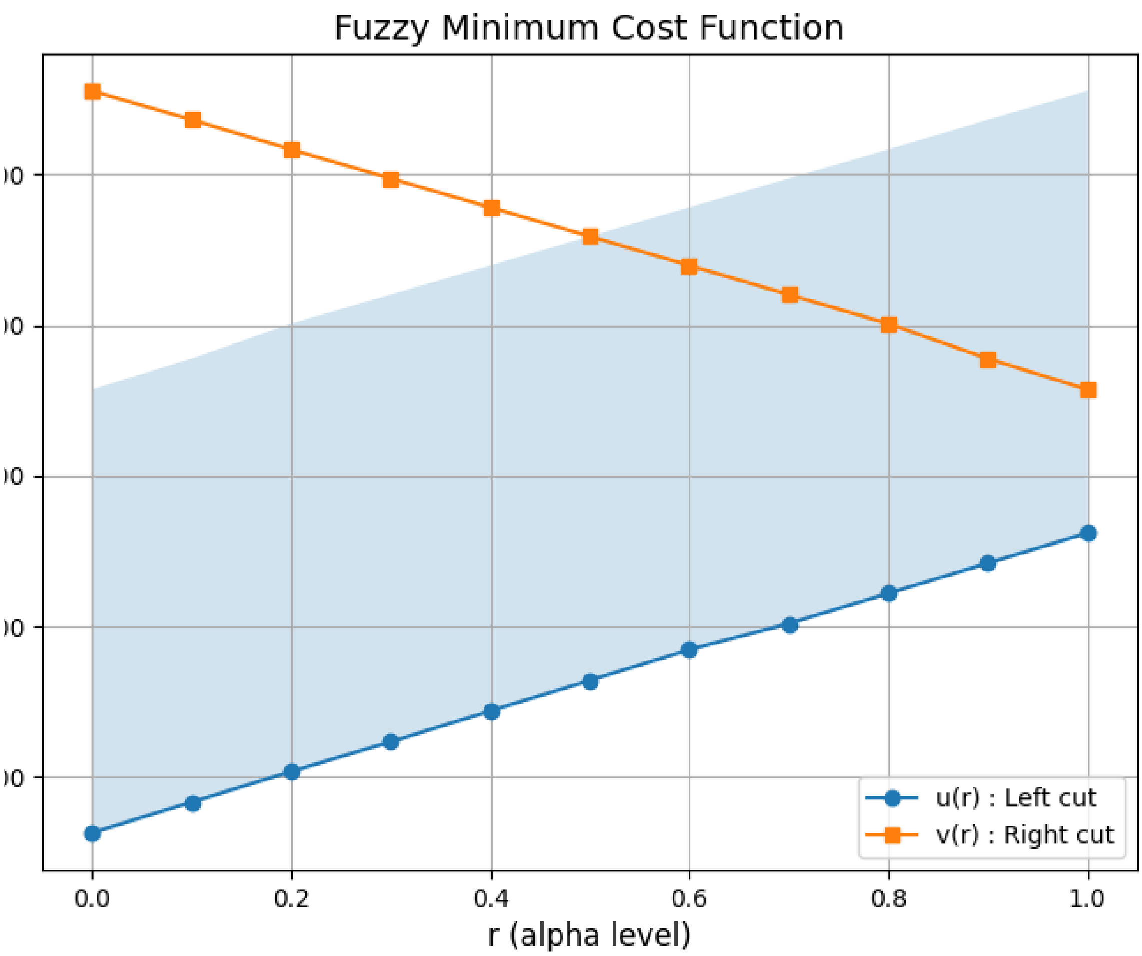

Figure 2: Variation of left and right end costs according to α levels

5.4 Comparative Analysis

The proposed method was compared with the classical Bellman-Ford-based minimum-cost flow algorithm. Experimental results show that the developed dynamic programming-based approach provides a lower solution time in multi-stage network structures.

It has been observed that the computational cost of classical methods increases rapidly, especially as the number of nodes increases, while the proposed method maintains its scalability.

The experimental findings reveal that the proposed algorithm produces stable solutions under different levels of uncertainty. Thanks to the capacity dominance criterion, selecting paths with higher flow capacity among paths with the same cost positively affects the overall performance.

Furthermore, the dynamic programming-based shortest path algorithm offers a significant computational advantage compared to classical approaches by enabling the efficient use of the multi-stage network structure.

6 Conclusions and Future Work

In this study, a new dynamic programming-based heuristic algorithm is proposed for solving the fuzzy minimum cost maximum flow problem defined on multi-stage networks. The proposed method transforms parametric fuzzy costs into deterministic subproblems using the α-section approach and produces efficient solutions by taking advantage of the staged structure of the network.

The developed algorithm provides lower computational costs compared to classical shortest path-based methods and obtains more balanced growth paths thanks to the capacity dominance criterion. Computational experiments have shown that the method produces stable, consistent, and scalable solutions under different levels of uncertainty.

The results indicate that the proposed approach can be effectively applied, especially in layered structures such as multi-stage supply chain networks. Furthermore, parametric modeling of fuzzy costs ensures a more realistic reflection of uncertainty in the decision-making process.

Future studies plan to improve the proposed method in the following areas:

- Integration of network models containing fixed charges into the algorithm,
- Modeling capacity values in a fuzzy manner,
- Addition of environmental and temporal criteria within the framework of multi-objective optimization,
- Implementation of applications on large-scale real datasets,
- Further reduction of solution time using parallel and distributed computing techniques.

Accordingly, it is believed that the proposed approach will form an important foundation for future studies in the field of fuzzy network optimization.

References

[1] Abhijit Baidya. Application of grey number to solve multi-stage supply chain networking model. *International Journal of Logistics Systems and Management*, 47(4):494–518, 2024.

[2] Alexander Bozhenyuk and Evgeniya Gerasimenko. Algorithm for monitoring minimum cost in fuzzy dynamic networks. *Information Technology and Management Science*, 16(1):53–59, 2013.

[3] Alexander Bozhenyuk, Evgeniya Gerasimenko, Igor Rozenberg, and Irina Perfilieva. Method for the minimum cost maximum flow determining in fuzzy dynamic network with nonzero flow bounds. In *Proceedings of the 2015 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology*, pages 385–392. Atlantis Press, 2015.

[4] Li Chen, Rasmus Kyng, Yang Liu, Richard Peng, Maximilian Probst Gutenberg, and Sushant Sachdeva. Maximum flow and minimum-cost flow in almost-linear time. *Journal of the ACM*, 72(3):1–103, 2025.

[5] Li Chen, Rasmus Kyng, Yang P Liu, Richard Peng, Maximilian Probst Gutenberg, and Sushant Sachdeva. Almost-linear-time algorithms for maximum flow and minimum-cost flow. *Communications of the ACM*, 66(12):85–92, 2023.

[6] Fu-Lai Chung and Ji-Cheng Duan. On multistage fuzzy neural network modeling. *IEEE Transactions on fuzzy systems*, 8(2):125–142, 2000.

[7] Jie Ding, Changyun Wen, Guoqi Li, Pengfei Tu, Dongxu Ji, Ying Zou, and Jiangshuai Huang. Target controllability in multilayer networks via minimum-cost maximum-flow method. *IEEE Transactions on Neural Networks and Learning Systems*, 32(5):1949–1962, 2021.

[8] Nasser A. El-Sherbeny. The fuzzy minimum cost flow problem with the fuzzy time-windows. *American Journal of Applied Mathematics and Statistics*, 7(6):191–195, 2019.

[9] Nizami A Gasilov and Şahin Emrah Amrahov. A survey and comparative analysis of different approaches to fuzzy differential equations modeling dynamics with uncertain parameters of deterministic character. *Kybernetika*, 61(3):289–347, 2025.

[10] Mehdi Ghatee and S Mehdi Hashemi. Some concepts of the fuzzy multicommodity flow problem and their application in fuzzy network design. *Mathematical and Computer Modelling*, 49(5–6):1030–1043, 2009.

[11] Nermin Kartli, Erkan Bostanci, and Mehmet Serdar Guzel. Heuristic algorithm for an optimal solution of fully fuzzy transportation problem. *Computing*, 106(10):3195–3227, 2024.

[12] Wang Keyi and Wang Zhongtuo. Fuzzy minimum-cost flow in network and its application in transportation problems. In *Control Science and Technology for Development*, pages 407–410. Elsevier, 1986.

[13] Milan Mares and Josef Horák. Fuzzy quantities in networks. *Fuzzy Sets and Systems*, 10(1-3):123–134, 1983.

[14] Xiaodong Shi, Xiangping Zhai, Rui Wang, Yi Le, Shuang Fu, and Ningzhong Liu. Task planning of multiple unmanned aerial vehicles based on minimum cost and maximum flow. *Sensors*, 25(5):1605, 2025.